

Pion form factor in large N_c QCD*

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Abstract

The electromagnetic form factor of the pion is obtained using a particular realization of QCD in the large N_c limit, which sums up the infinite number of zero-width resonances to yield an Euler's Beta function of the Veneziano type. This form factor agrees with space-like data much better than single rho-meson dominance. A simple unitarization ansatz is illustrated, and the resulting vector spectral function in the time-like region is shown to be in reasonable agreement with the ALEPH data from threshold up to about 1.3 GeV².

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It is well known that QCD in the limit of large number of colours (QCD_∞) [1] predicts the existence of an infinity of zero-width resonances [2]. However, in the absence of exact solutions to this theory, the hadronic parameters (masses, widths, couplings) remain unspecified. Various models of this spectrum have been proposed for heavy quark Green's functions [3]-[4], as well as for light quark systems [5]. An infinite set of narrow resonances is reminiscent of Veneziano's dual model [6]; in fact, such a connection has been studied recently for the case of vector and axial-vector two-point functions [7]. In this note the three-point function determining the electromagnetic form factor of the pion is analyzed using a specific realization of QCD_∞ , namely a choice of masses and couplings that yields an Euler's Beta function of the Veneziano type. This realization will be called Dual- QCD_∞ in the sequel. Models of this kind have been studied long before the advent of QCD_∞ , in connection with electromagnetic form factors [8], purely hadronic three-point functions [9]-[10], as well as two-point functions [11]. However, the connection to QCD_∞ was lacking. The main purpose of this note is to show that the particular choice of masses and couplings leading to a Beta function form factor is a very viable way of modelling QCD_∞ . In fact, this form factor exhibits asymptotic Regge behaviour in the space-like region, where it agrees with experiment much better than single rho-meson dominance. Also, unlike four-point functions, this form factor can be unitarized without unwanted consequences. A simple unitarization ansatz will be illustrated. This leads to a smeared vector spectral function in the time-like region which can then be confronted, both locally and globally, with the ALEPH data [12].

With the standard definition of the pion form factor

$$\langle \pi(p_2) | J_\mu^{EM} | \pi(p_1) \rangle = (p_1 + p_2)_\mu F_\pi(s) , \quad (1)$$

where $s \equiv q^2 = (p_2 - p_1)^2$, one expects in QCD_∞

$$F_\pi(s) = \sum_{n=0}^{\infty} \frac{C_n}{(M_n^2 - s)} . \quad (2)$$

In order to obtain a Beta function type form factor the coefficients C_n are chosen as

$$C_n = \frac{\Gamma(\beta - 1/2)}{\alpha' \sqrt{\pi}} \frac{(-1)^n}{\Gamma(n+1)} \frac{1}{\Gamma(\beta - 1 - n)} , \quad (3)$$

where β is a free parameter to be fit e.g. from data in the space-like region ($s < 0$), and $\alpha' = 1/2M_\rho^2$ is the universal string tension entering the rho-meson Regge trajectory

$$\alpha_\rho(s) = 1 + \alpha'(s - M_\rho^2) . \quad (4)$$

On the other hand, the mass spectrum is chosen as [13]

$$M_n^2 = M_\rho^2(1 + 2n) . \quad (5)$$

These choices then lead to

$$\begin{aligned}
F_\pi(s) &= \frac{\Gamma(\beta - 1/2)}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \frac{1}{\Gamma(\beta - 1 - n)} \frac{1}{[n + 1 - \alpha_\rho(s)]} \\
&= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta - 1/2)}{\Gamma(\beta - 1)} B(\beta - 1, 1/2 - \alpha'_\rho s) ,
\end{aligned} \tag{6}$$

where $B(x,y)$ is Euler's Beta function, and $F_\pi(0) = 1$. In the time-like region ($s > 0$) the poles of the Beta function along the negative real axis correspond to an infinite set of zero-width resonances with equally spaced squared masses given by Eq.(5). Asymptotically, the pion form factor in the space-like region exhibits Regge behaviour, viz.

$$\lim_{s \rightarrow -\infty} F_\pi(s) = (-\alpha'_\rho s)^{(1-\beta)} , \tag{7}$$

which can be used to fix the parameter β from a fit to the data. Alternatively, β can be determined from the measured mean squared radius of the pion. As shown below, both procedures are nicely consistent. It should be noticed that the value $\beta = 2$ reduces the form factor to single rho-meson dominance (naive Vector Meson Dominance).

The mass formula Eq.(5) predicts e.g. for the first two radial excitations: $M_{\rho'} \simeq 1340$ MeV, and $M_{\rho''} \simeq 1720$ MeV, in reasonable agreement with experiment [14] : $M_{\rho'} = 1465 \pm 25$ MeV, and $M_{\rho''} = 1700 \pm 20$ MeV. The form factor Eq. (6) is shown in Fig. 1 (solid line), together with the available experimental data in the space-like region [15]-[16] (including the latest data from the Jefferson Lab), and for a least-squared fitted value $\beta \simeq 2.3$. The Vector Meson Dominance (VMD) prediction ($\beta = 2$) is also shown for comparison (dash line). The statistical significance of the difference between both predictions is illustrated by the resulting chi-squared values. In the case of $\beta = 2.3$, implying an infinite set of vector mesons, the chi-squared per degree of freedom is $\chi_F^2 \simeq 1.4$, while VMD ($\beta = 2$) gives $\chi_F^2 \simeq 11$, or an unacceptably poor fit to the data. On the other hand, the mean-squared radius which follows from Eq.(6) is $\langle r_\pi^2 \rangle = 0.436 \text{ fm}^2$, to be compared with the experimental value $\langle r_\pi^2 \rangle = 0.439 \pm 0.008 \text{ fm}^2$ [16], and that of VMD ($\beta = 2$): $\langle r_\pi^2 \rangle = 0.394 \text{ fm}^2$.

Dual- QCD_∞ may also be viewed as a particular realization of Extended VMD [13], according to which one expects in general

$$F_\pi(s) = \sum_{n=0}^{\infty} \frac{g_{\rho_n \pi \pi}}{f_n} \frac{M_n^2}{(M_n^2 - s)} , \tag{8}$$

where the f_n are the electromagnetic couplings of the photon to the vector mesons ρ_n of masses M_n , and $g_{\rho_n \pi \pi}$ are the $\rho_n \pi \pi$ strong couplings. In fact, in the framework of Dual- QCD_∞ , Eq.(6) may be rewritten

as

$$F_\pi(s) = \frac{g_{\rho\pi\pi}}{f_\rho} \frac{M_\rho^2}{(M_\rho^2 - s)} F_{\rho\pi\pi}(s) , \quad (9)$$

where

$$F_{\rho\pi\pi}(s) = \frac{\Gamma(\beta - 1)}{\Gamma(\beta - 2)} B(\beta - 2, 3/2 - \alpha' s) , \quad (10)$$

is the $\rho\pi\pi$ vertex function with an off-mass-shell rho-meson, normalized as $F_{\rho\pi\pi}(s = M_\rho^2) = 1$. Once again, for $\beta = 2$ one recovers the VMD result. From the normalization $F_\pi(0) = 1$, there follows the VMD universality relation

$$\frac{g_{\rho\pi\pi}}{f_\rho}|_{VMD} = 1 . \quad (11)$$

The disagreement between Eq.(11) and experiment:

$$\frac{g_{\rho\pi\pi}}{f_\rho}|_{EXP} = 1.21 \pm 0.02 , \quad (12)$$

provides strong support for the existence of radial excitations of the rho-meson. Furthermore, the prediction from Eqs.(9)-(10) for the fitted value $\beta \simeq 2.3$

$$\frac{g_{\rho\pi\pi}}{f_\rho}|_{QCD_\infty} = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\beta - 1/2)}{\Gamma(\beta - 1)} \simeq 1.2 , \quad (13)$$

provides additional strong support for Dual- QCD_∞ .

Turning to the time-like region ($s > 0$), clearly the zero-width feature of the spectrum in QCD_∞ is unrealistic. It is possible, though, to make the pion form factor in Dual- QCD_∞ compatible with unitarity. In fact, unitarization of Dual Model three-point functions does not lead to unwanted consequences [10],[17], as is the case with n-point functions ($n \geq 4$) [6]. It should be stressed that very little change in the behaviour of $F_\pi(s)$ in the space-like region is expected from smearing the spectrum in the time-like region. This process involves, in addition to the parameter β in Eq.(3), at least one more free parameter related to the widths of the radial excitations, as discussed next. The imaginary part of $F_\pi(s)$ which follows from Eq.(6) is

$$Im F_\pi(s) = \frac{\Gamma(\beta - 1/2)}{\alpha' \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \frac{1}{\Gamma(\beta - 1 - n)} \pi \delta(M_n^2 - s) . \quad (14)$$

A simple unitarization prescription is to make the substitution [10], [17]

$$\pi \delta(M_n^2 - s) \rightarrow \frac{\Gamma_n M_n}{[(M_n^2 - s)^2 + \Gamma_n^2 M_n^2]} . \quad (15)$$

This Breit-Wigner resonance spectrum is expected to be a reasonable approximation for relatively narrow resonances. In addition, the correct threshold behaviour of the imaginary part of $F_\pi(s)$ must be enforced; in the chiral $SU(2) \times SU(2)$ limit $Im F_\pi(s) \rightarrow s \rightarrow 0$. As to the n-dependence of the radial excitation widths, these are expected to grow with increasing mass. The following simple-minded ansatz will be

adopted for the purpose of illustration: $\Gamma_n = \gamma M_n$, with γ fixed by the rho-meson parameters ($\gamma \simeq 0.2$). Having thus specified the imaginary part of $F_\pi(s)$, its real part can be computed from the dispersion relation

$$\text{Re } F_\pi(s) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im } F_\pi(s')}{(s' - s)} ds' , \quad (16)$$

where the integral is to be understood as its principal value.

It has been checked that the nice agreement between $F_\pi(s)$ in Dual- QCD_∞ and the data for $s \leq 0$ remains essentially unchanged by unitarization. However, once smeared, $F_\pi(s)$ may be used to compute the vector spectral function defined as the imaginary part of the two-point function

$$\begin{aligned} \Pi_{\mu\nu}^{VV}(q^2) &= i \int d^4x e^{iqx} \langle 0 | T(V_\mu(x) V_\nu^\dagger(0)) | 0 \rangle \\ &= (-g_{\mu\nu} q^2 + q_\mu q_\nu) \Pi_V(q^2) , \end{aligned} \quad (17)$$

where $V_\mu(x) =: \bar{q}(x) \gamma_\mu q(x) :$, and $q = (u, d)$. In fact, using the normalization in which $\lim_{s \rightarrow \infty} \rho_V(s) = 1/2$ (at the one loop level in QCD), with $\rho_V(s) \equiv \frac{1}{\pi} \text{Im} \Pi_V(s)$, one has in the chiral limit

$$|F_\pi(s)|^2 = 12 \rho_V(s) . \quad (18)$$

Figure 2 shows the ALEPH data on the vector spectral function together with that computed from the unitarized pion form factor (solid curve). It is quite encouraging that despite the simplicity of the unitarization ansatz chosen here, the agreement with the data is quite reasonable up to about $s \simeq 1.3 \text{ GeV}^2$. Globally, the agreement is even better, viz. the area under the theoretical spectral function is

$$\int_0^{s_0} \rho_V(s) |_{QCD_\infty} ds = 1.1 \text{ GeV}^2 , \quad (19)$$

to be compared with the experimental value

$$\int_0^{s_0} \rho_V(s) |_{EXP} ds = 0.9 \text{ GeV}^2 , \quad (20)$$

where $s_0 = 1.25 \text{ GeV}^2$. For higher energies the Dual- QCD_∞ spectral function falls faster than the data. However, this is not a serious drawback, since at these energies one expects perturbative QCD to take over. In fact, standard models of the spectrum consist of hadronic resonance parametrizations up to s_0 , and from there onwards the perturbative QCD expression.

In summary, Dual- QCD_∞ provides a reasonable model for the spectrum of infinite zero-width vector-isovector resonances, to the extent that the resulting pion form factor is in very good agreement with experimental data in the space-like region, after fitting the single free parameter of the model (β in Eqs.

(3) and (6)). Moreover, the resulting pion mean squared radius, and the ratio of strong to electromagnetic couplings also agree very well with the data. In contrast, naive VMD is in poor agreement with experiment on all three counts. In the time-like region, the model can be easily unitarized. A very simple ansatz shows that good agreement with the experimental vector spectral function can be achieved, both locally and globally, up to about $s \simeq 1.3 \text{ GeV}^2$. This is normally sufficient, as for higher energies one usually assumes the spectral function to be given by perturbative QCD. Clearly, the unitarization ansatz discussed here is subject to considerable improvement.

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Figure Captions

Figure 1. Dual- QCD_∞ pion form factor, Eq.(6), in the space like region for the fitted parameter $\beta \simeq 2.3$ (solid curve), together with naive VMD ($\beta = 2$) (dash curve), and the experimental data [15]-[16].

Figure 2. Dual- QCD_∞ vector spectral function in the time-like region, from Eq.(18) after unitarization, together with the ALEPH data [12].

Figure 1

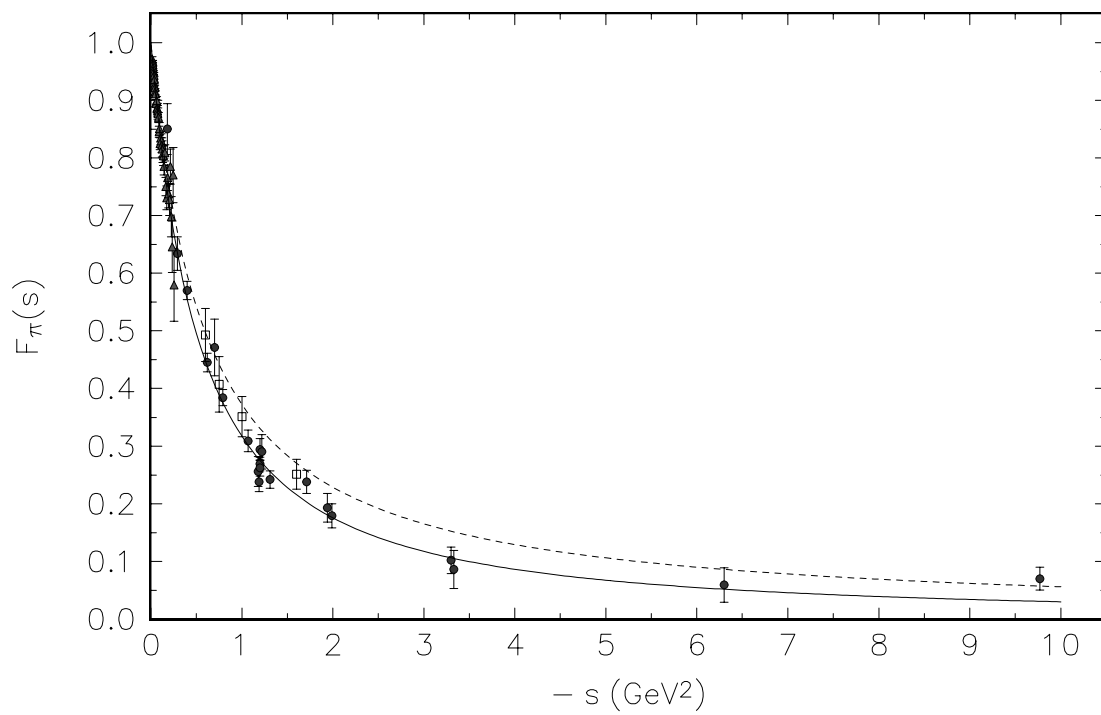


Figure 1:

Figure 2

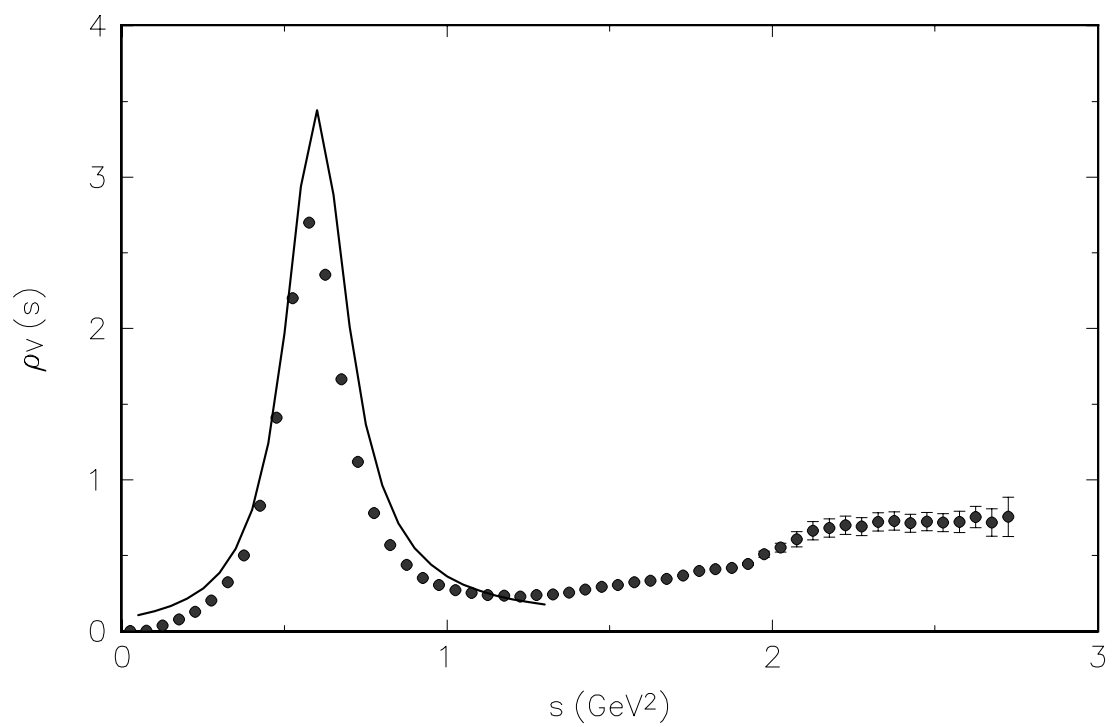


Figure 2: